Vertex Splitting and Upper Embeddable Graphs †

Guanghua Dong[1,2*], Ning Wang[3], Yuanqiu Huang[1], Han Ren[4], Yanpei Liu[5]

1.Department of Mathematics, Normal University of Hunan, Changsha, 410081, China

2.Department of Mathematics, Tianjin Polytechnic University, Tianjin, 300160, China

3.Department of Information Science and Technology, Tianjin University of Finance
   and Economics, Tianjin, 300222, China

4.Department of Mathematics, East China Normal University, Shanghai, 200062,China

5.Department of Mathematics, Beijing Jiaotong University, Beijing, 100044, China

Abstract

The *weak minor* $\underline{G}$ of a graph G is the graph obtained from G by a sequence of edge-contraction operations on G. A *weak-minor-closed* family of upper embeddable graphs is a set $\mathcal{G}$ of upper embeddable graphs that for each graph G in $\mathcal{G}$, every weak minor of G is also in $\mathcal{G}$. Up to now, there are few results providing the necessary and sufficient conditions for characterizing upper embeddability of graphs. In this paper, we studied the relation between the vertex splitting operation and the upper embeddability of graphs; provided not only a necessary and sufficient condition for characterizing upper embeddability of graphs, but also a way to construct weak-minor-closed family of upper embeddable graphs from the bouquet of circles; extended a result in *J. Graph Theory* obtained by L. Nebeský. In addition, the algorithm complex of determining the upper embeddability of a graph can be reduced much by the results obtained in this paper.

Key Words: maximum genus; weak minor; flexible-weak-minor; flexible-vertex; flexible-edge

MSC(2000): 05C10

1. Introduction

Graphs considered here are all connected, undirected, and with minimum degree at least three. In addition, multiple edges and loops are permitted. Terminologies and notations not defined here can be seen in [1]. The reader is assumed to be familiar with topological graph theory, which can be find more details in [2], [3] or [4].

A graph is denoted by $G = (V(G), E(G))$, and $V(G)$, $E(G)$ denotes its vertex set and edge set respectively. The number $|E(G)| - |V(G)| + 1$ is known as the *Betti number* (or *cycle rank*) of the connected graph G, and is denoted by $\beta(G)$. A u, v-*path* is a path whose vertices of degree 1 (its endpoints) are u and v. Let T be a spanning tree of a connected graph G. Define the *deficiency* $\xi(G, T)$ of a spanning tree T in a graph G to be the number of components of $G - E(T)$ which have odd size. The deficiency $\xi(G)$ of a graph G is defined to be the minimum value of $\xi(G, T)$ over all spanning tree T of G, *i.e.*, $\xi(G) = min\{\xi(G, T) \mid$ T is an spanning tree of G$\}$. A *splitting tree* of a connected graph G is a spanning tree T for G such that at most one component of $G - E(T)$ has odd size. Let v be a vertex of G, and $N_G(v)$ be the set of vertices in G adjacent to v, then the subgraph induced by $N_G(v)$ is referred to as the *v-local* subgraph, and is denoted by $G_{loc}(v)$. The *vertex splitting* on a vertex v, whose degree $\deg_G(v) \geqslant 4$, is the replacement of the vertex v by adjacent vertices v' and v'' and the replacement of each edge $e = vu$ incident to v either by the edge $v'u$ or by the edge $v''u$, and the edge $v'v''$ in the new G^* is called the *splitting-edge*. If G^* is a graph obtained from G by a vertex splitting operation on the vertex $v \in V(G)$, then the subgraph of G^*, which is induced by v', v'' and the vertices adjacent to v' and v'', is refereed to as the *v-splitting subgraph* and is denoted by $G^*_{spl}(v)$. The *intersection* of two graphs G_1 and G_2 is defined as $G_1 \cap G_2 = (V(G_1) \cap V(G_2), E(G_1) \cap E(G_2))$, and the *union* of G_1 and G_2 is defined as $G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. A *partial order* $\mathcal{R}$ on a set X is a binary relation that is reflexive, antisymmetric, and transitive. A *poset*, which is short for *partially ordered set*, is a pair $(X; \mathcal{R})$ where X is a set and $\mathcal{R}$ is a *partial order relation* on X. The *weak minor* $\underline{G}$ of a graph G, which is denoted by $\underline{G} \preccurlyeq G$, is the graph obtained from G by a sequence of edge-contraction operations on G. Furthermore, a graph G is a weak minor of itself. For example, both G_1 in Fig.2 and G_2 in Fig.3 are a weak-minor of the graph G in Fig.1. A *weak-minor-closed* family of upper embeddable graphs is a set $\mathcal{G}$ of upper embeddable graphs that for each graph G in $\mathcal{G}$, every weak minor of G is also in $\mathcal{G}$. Obviously, the binary relation *weak minor*, which is denoted by $\preccurlyeq$, is a *partial order*.
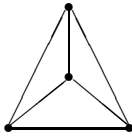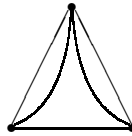

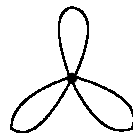**Fig.1**: G   **Fig.2**: G_1 **Fig.3**: G_2

The *maximum genus* $\gamma_M(G)$ of a connected graph G is the maximum integer k such that there exists an embedding of G into the orientable surface of genus k. A graph G is said to be *upper embeddable* if $\gamma_M(G) = \lfloor \frac{\beta(G)}{2} \rfloor$. Nordhaus, Stewart and White

[5] introduced the idea of the maximum genus of graphs in 1971. From then on, many interesting results have being made, mainly concerned with the relation between the maximum genus and other graph parameters as diameter, face size, connectivity, girth, etc., and the readers can find more details in [6][7][8][9][10][11][12][13][14][15] etc.. But few papers have provided the informations about the problems as: (I) the relation between the upper embeddability and vertex splitting; (II) the weak-minor-closed family of upper embeddable graphs. The following is the details for the two problems.

Problem I: Let G be an upper embeddable graph, v be a vertex of G with degree no less than 4, and G^* be the graph obtained from G through a vertex splitting operation on v, then G^* may be upper embeddable or not. For example, both the graph G_1 in Fig.5 and the graph G_2 in Fig.6 are obtained from an upper embeddable G in Fig.4 through a vertex splitting operation on v in G. The graph G_1 is upper embeddable, but G_2 is not upper embeddable. So, a question is naturally raised: How does an upper embeddable graph remain the upper embeddability after the vertex splitting operation on some vertex v of this graph?
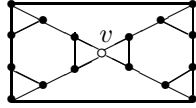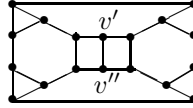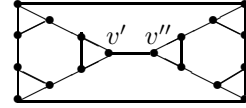

Fig.4: G   **Fig.5**: G_1 **Fig.6**: G_2

Problem II: In general, a class of upper embeddable graphs is not closed under minors. For example, although the graph G depicted in Fig.8 is upper embeddable, the graph G_1 in Fig.7, which is a minor of G, is not upper embeddable. But, if G is an upper embeddable graph then every weak minor $\underline{G}$ of G is also upper embeddable. So we can easily get a poset $\mathcal{F}$, which is a weak-minor closed family of upper embeddable graphs, from G through a sequence of edge-contraction operations on G. Obviously, the bouquet of circles $B_{\beta(G)}$, which consists of a single vertex with $\beta(G)$ loops incident to this vertex, is the smallest element of $\mathcal{F}$, $i.e.$, every upper embeddable graph with $\beta(G)$ co-tree edges has bouquet circles $B_{\beta(G)}$ as its weak-minor. However, from the example in Fig.4-Fig.6 we can get that the bouquet circles $B_{\beta(G)}$ may also be a weak-minor of a graph G which is not upper embeddable. So, how to get a poset $\mathcal{F}$, which is a weak-minor-closed family of upper embeddable graphs, from the bouquet of circles B_n or other upper embeddable graph via series of vertex-splitting operations on it is the second problem.
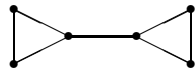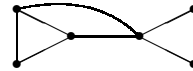

Fig.7: G_1 **Fig.8**: G

In this paper, we will do some research on the above two problems. The following is a Lemma which is obtained by Liu [4][16] and Xuong [15] independently.

Lemma 1.1   Let G be a connected graph, then
1) $\gamma_M(G) = \frac{\beta(G)-\xi(G)}{2}$;

2) G is upper embeddable if and only if $\xi(G) \leqslant 1$, or G has a splitting tree.

2. Vertex splitting and upper embeddability

As described in the introduction, an upper embeddable graph may be changed into a non-upper embeddable graph after a vertex splitting operation. How does a graph remain the upper embeddability after vertex splitting operations? In this section, we provide some results on this problem.

Lemma 2.1 Let G be an upper embeddable graph, v be a vertex of G with $\deg_G(v) \geqslant 3$, and $v_1, v_2, \ldots, v_n$ be all the neighbors of v in G. If the v-*local* subgraph $G_{loc}(v)$ is connected, then there must exist a splitting tree $\mathbb{T}$ of G such that all of $\{vv_1, vv_2, \ldots, vv_n\}$ are edges of $\mathbb{T}$.

Proof Let T be an arbitrary splitting tree of G. Since $v_1, v_2, \ldots, v_n$ are all the neighbors of v in G, the splitting tree T must contain at least one of $\{vv_i | i = 1, 2, \ldots, n\}$ as its edge. Without loss of generality, it may be assumed that $vv_1 \in E(T)$.

If each of $\{vv_i | i = 2, \ldots, n\}$ is an edge of T, then the splitting tree T is $\mathbb{T}$ itself.

If some edges of $\{vv_i | i = 2, \ldots, n\}$ are not in T, then assume, without loss of generality, that $vv_{i_1}, vv_{i_2}, \ldots, vv_{i_m} (m \leqslant n - 1)$ are all the edges of $\{vv_i | i = 2, \ldots, n\}$ which are not in T, where the vertex set $\{v_{i_1}, v_{i_2}, \ldots, v_{i_m}\} \subseteq \{v_2, \ldots, v_n\}$. Let v_{i_j} be an arbitrary vertex of $\{v_{i_1}, v_{i_2}, \ldots, v_{i_m}\}$. Because there is exactly one u, ω-*path* in T for any two vertices u and ω in G, and the edge vv_{i_j} is not in T, there must be a vv_{i_j}-*path* in T, and the vv_{i_j}-*path* in T must be the style: $v \ldots v_\alpha v_{i_j}$, where v_α is a vertex of $\{V(G) - \{v, v_{i_j}\}\}$. Let $T_{i_j} = \{T - v_\alpha v_{i_j}\} \cup vv_{i_j}$. It is obvious that T_{i_j} is a spanning tree of G and the edge $vv_{i_j} \in E(T_{i_j})$. Through series of processes similar to that of getting T_{i_j}, a spanning tree T^* is obtained, where all of $\{vv_1, vv_2, \ldots, vv_n\}$ are edges of T^*. Since all edges of $\{vv_1, vv_2, \ldots, vv_n\}$ are in T^*, each edge of $G_{loc}(v)$ is not in T^*, or else the spanning tree T^* will contain cycles. So all edges of $G_{loc}(v)$ are co-tree edges of T^*. Because the v-*local* subgraph $G_{loc}(v)$ is connected, we can get that $\xi(G, T^*) \leqslant \xi(G, T) = \xi(G) \leqslant 1$. So T^* is a splitting tree of G which satisfies the Lemma. $\quad\square$

Lemma 2.2 Let G be an upper embeddable graph with minimum degree at least 3, v be a vertex of G with $\deg_G(v) = 4$, G^* be the graph obtained from G by splitting v into two adjacent vertices v' and v''. If the *splitting-edge* $v'v''$ is not a cut-edge of the v-*splitting* subgraph $G^*_{spl}(v)$, then G^* is upper embeddable.

Proof Let v_1, v_2, v_3, v_4 be the four vertices adjacent to v in G, and $\mathbb{T}$ be a splitting tree of G. Since $v'v''$ is not a cut-edge of the v-*splitting* subgraph $G^*_{spl}(v)$, $G^*_{spl}(v)$ must contain at least one cycle which has $v'v''$ as one of its edges. Without loss of generality, let $v_{i_1} v_{i_2} v'' v'$ be the 4-cycle of $G^*_{spl}(v)$, which is depicted, for example, in Fig.9 or Fig.11, where $\{v_{i_1}, v_{i_2}\} = \{v_1, v_2\}$. Because G^* is obtained from G through vertex splitting operation on v, $v_1 v_2 v$ must be a 3-cycle of G, which is depicted, for example, in Fig.10. In graph G, let $C_i (i = 1, 2, 3, 4)$ denote the connected component which is obtained from such

connected component of $G - E(\mathbb{T})$ that contains v_i as one of its vertices, by deleting the edges $vv_1, vv_2, vv_3, vv_4, v_1v_2$ from it. It is possible that C_i and C_j may be the same connected component of $G - E(\mathbb{T})$ ($i, j = 1, 2, 3, 4$ and $i \neq j$). If G is upper embeddable, the graph G^* in Fig.11, which is obtained from G through vertex splitting on v, is upper embeddable, for G^* can also be viewed as a subdivision of G. So, we should only discuss the upper embeddability of G^* in Fig.9. For v_1, v_2, v_3, v_4 being all the neighbors of v in graph G, the splitting tree $\mathbb{T}$ of G must contain at least one edge which belongs to the edge set $E(v) = \{vv_i | i = 1, 2, 3, 4\}$. It will be discussed in three cases according to whether at least three edges of $E(v)$ are in $\mathbb{T}$, or exactly two edges of $E(v)$ are in $\mathbb{T}$, or only one edge of $E(v)$ is in $\mathbb{T}$. Without loss of generality, let the edges $v'v_{i_1}, v''v_{i_2}, v''v_3, v'v_4$ in G^* be the replacement of vv_1, vv_2, vv_3, vv_4 in G after vertex splitting on v, where the edge set $\{v'v_{i_1}, v''v_{i_2}\}$ may be $\{v'v_1, v''v_2\}$ or $\{v'v_2, v''v_1\}$.
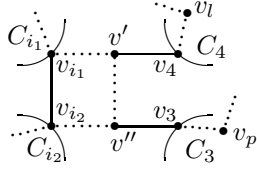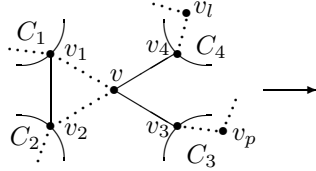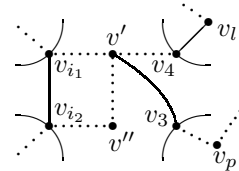

Fig.9: G^* **Fig.10:** G **Fig.11:** G^*

Case 1: At least three edges of $E(v)$ are in $\mathbb{T}$.

Without loss of generality, let vv_1, vv_2, ..., vv_n ($n = 3$ or 4) be all the edges of $E(v)$ which are in $\mathbb{T}$. Obviously, if exactly three edges of $E(v)$, which are denoted by $E_3(v)$, are in $\mathbb{T}$, and $E_3^*(v)$ denotes the replacement of $E_3(v)$ after vertex splitting on v in G, then $T^* = (G^* \cap \mathbb{T}) \cup v'v'' \cup E_3^*(v)$ is a spanning tree of G^*. If the four edges of $E(v)$ are all in $\mathbb{T}$, $T^* = (G^* \cap \mathbb{T}) \cup v'v'' \cup \{v'v_{i_1}, v''v_{i_2}, v''v_3, v'v_4\}$ is a spanning tree of G^*. Furthermore, $\xi(G^*, T^*) = \xi(G, \mathbb{T}) = \xi(G) \leqslant 1$. So T^* is a splitting tree of G^*, and in Case 1 G^* is upper embeddable.

Case 2: Exactly two edges of $E(v)$ are in $\mathbb{T}$.

The two edges of $E(v)$ in $\mathbb{T}$ may be (i) vv_1 and vv_2; or (ii) vv_3 and vv_4; or (iii) one edge belongs to $\{vv_1, vv_2\}$ and the other belongs to $\{vv_3, vv_4\}$.

Subcase 2.1: The two edges of $E(v)$ in $\mathbb{T}$ are vv_1 and vv_2.

In this case, the edge v_1v_2 in G can not be an edge of $\mathbb{T}$, or else vv_1v_2 would form a 3-cycle of $\mathbb{T}$. Let G^*, which is depicted in Fig.9, denotes the graph obtained from G through vertex splitting on v, where $\{C_{i_1}, C_{i_2}\} = \{C_1, C_2\}$, and $\{v_{i_1}, v_{i_2}\} = \{v_1, v_2\}$.

Subcase 2.1.1: C_3 and C_4 are the same connected component of G.

In this case, let $T^* = (G^* \cap \mathbb{T}) \cup v'v'' \cup \{v'v_{i_1}, v''v_{i_2}\}$. It is obvious that T^* is a spanning tree of G^*, and $\xi(G^*, T^*) = \xi(G, \mathbb{T}) = \xi(G) \leqslant 1$. So T^* is a splitting tree of G^*, and G^* is upper embeddable in Subcase-2.1.1.

5

Subcase 2.1.2: C_3 and C_4 are two different connected components of G.

In graph G^*, if at least one of $C_3 \cup v''v_3$ and $C_4 \cup v'v_4$ contains an even number of edges, then let $T^* = (G^* \cap \mathbb{T}) \cup v'v'' \cup \{v'v_{i_1}, v''v_{i_2}\}$. It is obvious that $\xi(G^*, T^*) = \xi(G, \mathbb{T}) = \xi(G) \leqslant 1$. So T^* is a splitting tree of G^*, and G^* is upper embeddable.

If both $C_3 \cup v''v_3$ and $C_4 \cup v'v_4$ contain an odd number of edges, then C_3 and C_4 both contain an even number of edges. Because there is exactly one u, ω-*path* in $\mathbb{T}$ for any two vertices u and ω in G, and both vv_3 and vv_4 are not in $\mathbb{T}$, there must be exactly one v, v_3-*path* in $\mathbb{T}$, and the v, v_3-*path* in $\mathbb{T}$ must be of the form as $vv_1 \dots v_p v_3$ or $vv_2 \dots v_p v_3$. Also, there must be exactly one v, v_4-*path* in $\mathbb{T}$, and the v, v_4-*path* in $\mathbb{T}$ must be of the form as $vv_1 \dots v_l v_4$ or $vv_2 \dots v_l v_4$. Furthermore, the v, v_3-*path* and v, v_4-*path* in $\mathbb{T}$ can not form a cycle. It is discussed in the following three subcases.

Subcase 2.1.2-a: The v, v_3-*path* and v, v_4-*path* in $\mathbb{T}$ are $vv_1 \dots v_p v_3$ and $vv_1 \dots v_l v_4$ respectively.

If the edges vv_1 and vv_2 in G are replaced, after the vertex splitting on v, by $v'v_{i_1}$ and $v''v_{i_2}$ respectively, then $T_1^* = (G^* \cap \mathbb{T}) \cup \{v'v_4, v''v_3, v''v_{i_2}\}$ is a spanning tree of G^*. Noticing that the size of $C_{i_1} \cup v_{i_1}v_{i_2} \cup C_{i_2} \cup v_{i_1}v' \cup v'v''$ and $C_{i_1} \cup v_{i_1}v_{i_2} \cup C_{i_2}$ have the same parity, and both the size of C_3 and C_4 are an even number, we can easily get that $\xi(G^*, T_1^*) = \xi(G, \mathbb{T}) = \xi(G) \leqslant 1$. So T_1^* is a splitting tree of G^*, and G^* is upper embeddable.

After the vertex splitting on v in G, if the edge vv_1 is replaced by $v''v_{i_2}$, and vv_2 by $v'v_{i_1}$ respectively, then $T_2^* = (G^* \cap \mathbb{T}) \cup \{v'v_4, v'v_{i_1}, v''v_3\}$ is a spanning tree of G^*. It is obvious that $\xi(G^*, T_2^*) = \xi(G, \mathbb{T}) = \xi(G) \leqslant 1$. So T_2^* is a splitting tree of G^*, and G^* is upper embeddable.

Subcase 2.1.2-b: The v, v_3-*path* and v, v_4-*path* in $\mathbb{T}$ are $vv_2 \dots v_p v_3$ and $vv_1 \dots v_l v_4$ respectively.

In this case, let $T^* = (G^* \cap \mathbb{T}) \cup \{v'v_4, v'v'', v''v_3\}$ be a spanning tree of G^*. It is obvious that $\xi(G^*, T^*) = \xi(G, \mathbb{T}) = \xi(G) \leqslant 1$. So T^* is a splitting tree of G^*, and G^* is upper embeddable.

Subcase 2.1.2-c: The v, v_3-*path* and v, v_4-*path* in $\mathbb{T}$ are $vv_2 \dots v_p v_3$ and $vv_2 \dots v_l v_4$ respectively, or $vv_1 \dots v_p v_3$ and $vv_2 \dots v_l v_4$ respectively.

In this case, it is similar to that of Subcase 2.1.2-a and Subcase 2.1.2-b to get that G^* contains a splitting tree.

So, in Subcase-2.1.2, G^* is upper embeddable.

Subcase 2.2: The two edges of $E(v)$ in $\mathbb{T}$ are vv_3 and vv_4.

In this case, according to v_1v_2 being an edge of $\mathbb{T}$ or not, it will be discussed in the following two subcases.

Subcase 2.2.1: The edge v_1v_2 of G is not in $\mathbb{T}$.

In this case, let $T^* = (G^* \cap \mathbb{T}) \cup \{v'v_4, v'v'', v''v_3\}$ be a spanning tree of G^*. It is obvious that $\xi(G^*, T^*) = \xi(G, \mathbb{T}) = \xi(G) \leqslant 1$. So T^* is a splitting tree of G^*.

Subcase 2.2.2: The edge v_1v_2 of G is an edge of $\mathbb{T}$.

It will be discussed in the following subcases.

Subcase 2.2.2-1: C_{i_1} and C_{i_2} are the same connected component of G.

In this case, let $T^* = (G^* \cap \mathbb{T}) \cup \{v'v_4, v'v'', v''v_3\}$ be a spanning tree of G^*. It is obvious that $\xi(G^*, T^*) = \xi(G, \mathbb{T}) = \xi(G) \leqslant 1$. So T^* is a splitting tree of G^*.

Subcase 2.2.2-2: C_{i_1} and C_{i_2} are two different connected components of G.

If at least one of $C_{i_1} \cup v'v_{i_1}$ and $C_{i_2} \cup v''v_{i_2}$ contains an even number of edges, then let $T^* = (G^* \cap \mathbb{T}) \cup \{v'v_4, v'v'', v''v_3\}$. It is obvious that $\xi(G^*, T^*) = \xi(G, \mathbb{T}) = \xi(G) \leqslant 1$. So T^* is a splitting tree of G^*, and G^* is upper embeddable.

If both $C_{i_1} \cup v'v_{i_1}$ and $C_{i_2} \cup v''v_{i_2}$ contain an odd number of edges, then C_{i_1} and C_{i_2} both contain an even number of edges. Because there is exactly one u, ω-path in $\mathbb{T}$ for any two vertices u and ω in G, and both vv_1 and vv_2 are not in $\mathbb{T}$, there must be exactly one v, v_1-path in $\mathbb{T}$, and this v, v_1-path in $\mathbb{T}$ may be the form as $vv_4 \ldots v_1v_2$, or $vv_4 \ldots v_2v_1$, or $vv_3 \ldots v_1v_2$, or $vv_3 \ldots v_2v_1$. It is discussed in the following two subcases.

Subcase 2.2.2-2a: The v, v_1-path in $\mathbb{T}$ is $vv_4 \ldots v_1v_2$ or $vv_4 \ldots v_2v_1$.

In this case, let $T^* = (G^* \cap \mathbb{T}) \cup \{v'v_4, v''v_3, v''v_{i_2}\}$. Noticing that both $C_{i_1} \cup v_{i_1}v' \cup v'v''$ and C_{i_2} contain an even number of edges, we can get that $\xi(G^*, T^*) = \xi(G, \mathbb{T}) = \xi(G) \leqslant 1$. So T^* is a splitting tree of G^*, and G^* is upper embeddable.

Subcase 2.2.2-2b: The v, v_1-path in $\mathbb{T}$ is $vv_3 \ldots v_1v_2$ or $vv_3 \ldots v_2v_1$.

In this case, let $T^* = (G^* \cap \mathbb{T}) \cup \{v'v_{i_1}, v'v_4, v''v_3\}$. It is obvious that $\xi(G^*, T^*) = \xi(G, \mathbb{T}) = \xi(G) \leqslant 1$. So T^* is a splitting tree of G^*, and G^* is upper embeddable.

Subcase 2.3: The two edges of $E(v)$ in $\mathbb{T}$ are such two edges that one is selected from $\{vv_1, vv_2,\}$ and the other is selected from $\{vv_3, vv_4\}$.

Without loss of generality, let the two edges of $E(v)$ in $\mathbb{T}$ are vv_1 and vv_3, which is illustrated in Fig.13. We will discuss in the following two subcases.

Subcase 2.3.1: After the vertex splitting on v in G, the replacements of vv_1 and vv_3 are both adjacent to v' or both adjacent to v''.

Without loss of generality, let the replacements of vv_1 and vv_3 are both adjacent to v'', which is illustrated in Fig.12. Let $T^* = (G^* \cap \mathbb{T}) \cup \{v''v_3, v''v_{i_2}, v'v''\}$. It is obvious that $\xi(G^*, T^*) = \xi(G, \mathbb{T}) = \xi(G) \leqslant 1$. So T^* is a splitting tree of G^*.
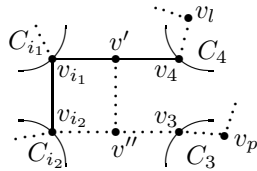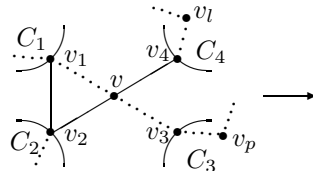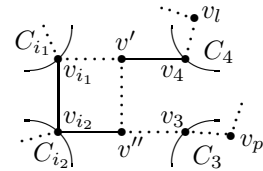

Fig.12: G^* **Fig.13:** G **Fig.14:** G^*

Subcase 2.3.2: After the vertex splitting on v in G, the replacements of vv_1 and vv_3 are adjacent to v' and v'' respectively.

7

Without loss of generality, let vv_1 and vv_3 be replaced, after vertex splitting on v, by $v'v_{i_1}$ and $v''v_3$ respectively, which is illustrated in Fig.14.

Subcase 2.3.2-1: In graph G, the edge v_1v_2 is not an edge of $\mathbb{T}$.

If C_4 and one of $\{C_{i_1}, C_{i_2}\}$ are the same connected component of G, then $T_1^* = (G^* \cap \mathbb{T}) \cup \{v'v_{i_1}, v'v'', v''v_3\}$ is a splitting tree of G^*.

If C_4 is a connected component of G which is different from both of $\{C_{i_1}, C_{i_2}\}$, we will discuss in two subcases.

Subcase 2.3.2-1a: At least one of $C_{i_1} \cup v_{i_1}v_{i_2} \cup C_{i_2} \cup v_{i_2}v''$ and $C_4 \cup v'v_4$ contains an even number of edges.

In this case, let $T^* = (G^* \cap \mathbb{T}) \cup \{v'v_{i_1}, v'v'', v''v_3\}$. It is obvious that $\xi(G^*, T^*) = \xi(G, \mathbb{T}) = \xi(G) \leqslant 1$. So T^* is a splitting tree of G^*, and G^* is upper embeddable.

Subcase 2.3.2-1b: Both $C_{i_1} \cup v_{i_1}v_{i_2} \cup C_{i_2} \cup v_{i_2}v''$ and $C_4 \cup v'v_4$ contain an odd number of edges.

In this case, C_4 contains an even number of edges. Because there is exactly one u, ω-path in $\mathbb{T}$ for any two vertices u and ω in G, and both vv_2 and vv_4 are not in $\mathbb{T}$, there must be exactly one v, v_4-path in $\mathbb{T}$, and the v, v_4-path in $\mathbb{T}$ must be the form as $vv_1 \ldots v_4$ or $vv_3 \ldots v_4$. If the v, v_4-path in $\mathbb{T}$ is $vv_1 \ldots v_4$, then $T_1^* = (G^* \cap \mathbb{T}) \cup \{v'v_4, v'v'', v''v_3\}$ is a splitting tree of G^*. If the v, v_4-path in $\mathbb{T}$ is $vv_3 \ldots v_4$, then $T_2^* = (G^* \cap \mathbb{T}) \cup \{v''v_3, v'v_4, v'v_{i_1}\}$ is a splitting tree of G^*.

Subcase 2.3.2-2: In graph G, the edge v_1v_2 is an edge of $\mathbb{T}$.

If at least one of $C_{i_2} \cup v''v_{i_2}$ and $C_4 \cup v'v_4$ contains an even number of edges, then let $T_1^* = (G^* \cap \mathbb{T}) \cup \{v'v_{i_1}, v'v'', v''v_3\}$. It is obvious that $\xi(G^*, T_1^*) = \xi(G, \mathbb{T}) = \xi(G) \leqslant 1$. So T_1^* is a splitting tree of G^*, and G^* is upper embeddable.

If both $C_{i_2} \cup v''v_{i_2}$ and $C_4 \cup v'v_4$ contain an odd number of edges, then C_{i_2} and C_4 both contain an even number of edges. Let $T_2^* = (G^* \cap \mathbb{T}) \cup \{v'v_{i_1}, v''v_{i_2}, v''v_3\}$. It is obvious that $\xi(G^*, T_2^*) = \xi(G, \mathbb{T}) = \xi(G) \leqslant 1$. So T_2^* is a splitting tree of G^*, and G^* is upper embeddable.

Case 3: Only one edge of $E(v)$ is in $\mathbb{T}$.

According to this edge is selected from $\{vv_1, vv_2\}$ or $\{vv_3, vv_4\}$, it will be discussed in the following Subcase-3.1 and Subcase-3.2.

Subcase 3.1: One of $\{vv_1, vv_2\}$ is the edge in $\mathbb{T}$.

Without loss of generality, let vv_1 be the edge in $\mathbb{T}$, which is depicted in Fig.16. In addition, throughout Subcase 3.1, let vv_1 and vv_2 be replaced by $v'v_1$ and $v''v_2$ respectively after the vertex splitting on v in G; and the edge set $\{vv_3, vv_4\}$ be replaced by $\{v''v_{i_3}, v'v_{i_4}\}$, where $\{v_{i_3}, v_{i_4}\} = \{v_3, v_4\}$ and $\{C_{i_3}, C_{i_4}\} = \{C_3, C_4\}$, which is depicted in Fig.15. According to the edge v_1v_2 of G is in the splitting tree $\mathbb{T}$ or not, it will be discussed in the following two subcases.
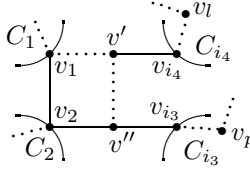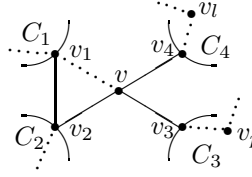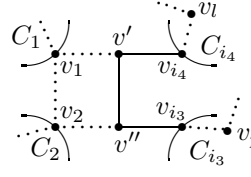
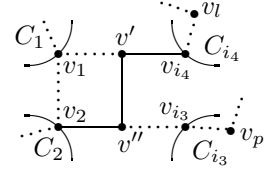**Fig.15:** G^* **Fig.16:** G **Fig.17:** G^* **Fig.18:** G^*

Subcase 3.1.1: In graph G, v_1v_2 is not an edge of $\mathbb{T}$. It is discussed in the following subcases.

Subcase 3.1.1-1: In graph G^*, $C_1 \cup v_1v_2 \cup C_2 \cup v_2v'' \cup v''v_{i_3} \cup C_{i_3} \cup C_{i_4} \cup v'v_{i_4}$ contains an odd number of edges.

In this case, $T^* = (G^* \cap \mathbb{T}) \cup \{v'v_1, v'v''\}$ is a splitting tree of G^*. So, in Subcase 3.1.1-1, G^* is upper embeddable.

Subcase 3.1.1-2: In graph G^*, $C_1 \cup v_1v_2 \cup C_2 \cup v_2v'' \cup v''v_{i_3} \cup C_{i_3} \cup C_{i_4} \cup v'v_{i_4}$ contains an even number of edges.

In this case, if $C_{i_4} \cup v'v_{i_4}$ contains an even number of edges, then $T^* = (G^* \cap \mathbb{T}) \cup \{v'v_1, v'v''\}$ is a splitting tree of G^*.

If $C_{i_4} \cup v'v_{i_4}$ contains an odd number of edges, then $C_1 \cup v_1v_2 \cup C_2 \cup v_2v'' \cup v''v_{i_3} \cup C_{i_3}$ contains an odd number of edges too. It is discussed in the following two subcases.

Subcase 3.1.1-2a: In graph G^*, the connected component C_{i_4} is the same with at least one of $\{C_1, C_{i_3}\}$.

In this case, $T^* = (G^* \cap \mathbb{T}) \cup \{v'v_1, v'v''\}$ is a splitting tree of G^*.

Subcase 3.1.1-2b: In graph G^*, neither of $\{C_1, C_{i_3}\}$ is the same connected component with C_{i_4}.

Because there is exactly one u, ω-*path* in $\mathbb{T}$ for any two vertices u and ω in G, and none of $\{vv_2, vv_3, vv_4\}$ is an edge of $\mathbb{T}$, there must be exactly one v, v_3-*path* and exactly one v, v_4-*path* in $\mathbb{T}$, and the v, v_3-*path*, v, v_4-*path* in $\mathbb{T}$ must be of the form as $vv_1 \ldots v_3$ and $vv_1 \ldots v_4$ respectively. Noticing that both C_{i_4} and $v'v_1 \cup C_1 \cup v_1v_2 \cup C_2 \cup v_2v'' \cup v''v_{i_3} \cup C_{i_3}$ are connected component of G^* with an even number of edges, we can easily get that $T^* = (G^* \cap \mathbb{T}) \cup \{v'v_{i_4}, v'v''\}$ is a splitting tree of G^*.

Subcase 3.1.2: In graph G, v_1v_2 is an edge of $\mathbb{T}$. It is discussed in the following subcases.

In graph G^*, if C_{i_4} is the same connected component with at least one of $\{C_1, C_2, C_{i_3}\}$, then $T^* = (G^* \cap \mathbb{T}) \cup \{v'v_1, v'v''\}$ is a splitting tree of G^*. If any pair of components, which is selected from $\{C_1, C_2, C_{i_3}, C_{i_4}\}$, is not the same connected component of G^*, then it will be discussed in the following two subcases.

Subcase 3.1.2-1: In graph G^*, $C_2 \cup v_2v'' \cup v''v_{i_3} \cup C_{i_3} \cup C_{i_4} \cup v'v_{i_4}$ contains an odd number of edges.

Noticing that one of $\{C_{i_4} \cup v'v_{i_4}, \ C_2 \cup v_2v'' \cup v''v_{i_3} \cup C_{i_3}\}$ is a connected component of G^* which contains an even number of edges, and the other is one which contains an odd number of edges, we can easily get that $T^* = (G^* \cap \mathbb{T}) \cup \{v'v_1, v'v''\}$ is a splitting tree of G^*.

9

Subcase 3.1.2-2: In graph G^*, $C_2 \cup v_2 v'' \cup v'' v_{i_3} \cup C_{i_3} \cup C_{i_4} \cup v' v_{i_4}$ contains an even number of edges.

If both $C_{i_4} \cup v' v_{i_4}$ and $C_2 \cup v_2 v'' \cup v'' v_{i_3} \cup C_{i_3}$ are connected component of G^* which contain an even number of edges, then it is easy to get that $T^* = (G^* \cap \mathbb{T}) \cup \{v' v_1, v' v''\}$ is a splitting tree of G^*.

If both $C_{i_4} \cup v' v_{i_4}$ and $C_2 \cup v_2 v'' \cup v'' v_{i_3} \cup C_{i_3}$ are connected component of G^* which contain an odd number of edges, then we will discuss it in the following two subcases.

Subcase 3.1.2-2a: In graph G^*, C_2 is a connected component with an even number of edges, and C_{i_3} is one which contains an odd number of edges.

Noticing that both C_2 and $C_{i_3} \cup v_{i_3} v'' \cup v'' v' \cup v' v_{i_4} \cup C_{i_4}$ are connected component of G^* which contain an even number of edges, we can easily get that $T^* = (G^* \cap \mathbb{T}) \cup \{v' v_1, v_2 v''\}$ is a splitting tree of G^*, which is depicted in Fig.17.

Subcase 3.1.2-2b: In graph G^*, C_2 is a connected component with an odd number of edges, and C_{i_3} is one which contains an even number of edges.

Because there is exactly one $u, \omega\text{-}path$ in $\mathbb{T}$ for any two vertices u and ω in G, and none of $\{vv_2, vv_3, vv_4\}$ is an edge of $\mathbb{T}$, there must be exactly one $v, v_3\text{-}path$ and exactly one $v, v_4\text{-}path$ in $\mathbb{T}$, and the $v, v_3\text{-}path$, $v, v_4\text{-}path$ in $\mathbb{T}$ must be of the form as $vv_1 \ldots v_3$ and $vv_1 \ldots v_4$ respectively. Noticing that, in the graph G^*, the connected components C_{i_3} and $C_2 \cup v_2 v'' \cup v'' v' \cup v' v_{i_4} \cup C_{i_4}$ both contain an even number of edges, we can easily get that $T^* = (G^* \cap \mathbb{T}) \cup \{v' v_1, v'' v_{i_3}\}$ is a splitting tree of G^*, which is depicted in Fig.18.

Subcase 3.2: One of $\{vv_3, vv_4\}$ is the edge in $\mathbb{T}$.

Without loss of generality, let vv_4 be the edge in $\mathbb{T}$, which is depicted in Fig.20. In addition, throughout Subcase 3.2, let vv_3 and vv_4 be replaced by $v'' v_3$ and $v' v_4$ respectively after the vertex splitting on v in G; and the edge set $\{vv_1, vv_2\}$ be replaced by $\{v' v_{i_1}, v'' v_{i_2}\}$, where $\{v_{i_1}, v_{i_2}\}=\{v_1, v_2\}$ and $\{C_{i_1}, C_{i_2}\}=\{C_1, C_2\}$, which is depicted in Fig.19. According to the edge $v_1 v_2$ of G is in the splitting tree $\mathbb{T}$ or not, it will be discussed in the following two subcases.
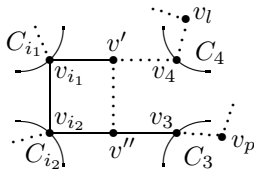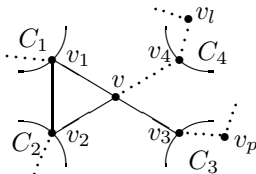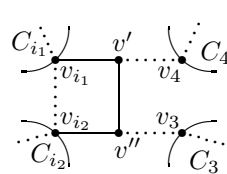

Fig.19: G^* **Fig.20:** G **Fig.21:** G^* **Fig.22:** G^*
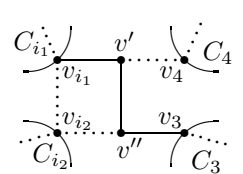
Subcase 3.2.1: In graph G, $v_1 v_2$ is not an edge of $\mathbb{T}$.

In this case, it is obvious that $T^* = (G^* \cap \mathbb{T}) \cup \{v' v_4, v' v''\}$ is a splitting tree of G^*, which is depicted in Fig.19. So, in Subcase 3.2.1, G^* is upper embeddable.

Subcase 3.2.2: In graph G, $v_1 v_2$ is an edge of $\mathbb{T}$.

In this case, if C_{i_1} in G^* is the same connected component with at least one of $\{C_{i_2}, C_3, C_4\}$, then $T^* = (G^* \cap \mathbb{T}) \cup \{v' v_4, v' v''\}$ is a splitting tree of G^*. If any pair

10

of components, which is selected from $\{C_{i_1}, C_{i_2}, C_3, C_4\}$, is not the same connected component of G^*, then it will be discussed in the following two subcases.

Subcase 3.2.2-1: In graph G^*, $C_{i_1} \cup v'v_{i_1}$ contains an even number of edges.

In this case, it is obvious that $T^* = (G^* \cap \mathbb{T}) \cup \{v'v_4, v'v''\}$ is a splitting tree of G^*, So, in Subcase 3.2.2-1, G^* is upper embeddable.

Subcase 3.2.2-2: In graph G^*, $C_{i_1} \cup v'v_{i_1}$ contains an odd number of edges, and $C_{i_2} \cup v_{i_2}v'' \cup v''v_3 \cup C_3$ contains an even number of edges.

In this case, the connected component $C_1 \cup v_1v \cup C_2 \cup v_2v \cup vv_3 \cup C_3$, which contains an odd number of edges in G, is replaced by $C_{i_1} \cup v'v_{i_1}$ and $C_{i_2} \cup v_{i_2}v'' \cup v''v_3 \cup C_3$ after the vertex splitting on v in G. Let $T^* = (G^* \cap \mathbb{T}) \cup \{v'v_4, v'v''\}$. Because $C_{i_1} \cup v'v_{i_1}$ contains an odd number of edges, and $C_{i_2} \cup v_{i_2}v'' \cup v''v_3 \cup C_3$ contains an even number of edges, it is obvious that $\xi(G^*, T^*) = \xi(G, \mathbb{T}) \leqslant 1$. So, T^* is a splitting tree of G^*.

Subcase 3.2.2-3: In graph G^*, both $C_{i_1} \cup v'v_{i_1}$ and $C_{i_2} \cup v_{i_2}v'' \cup v''v_3 \cup C_3$ contain an odd number of edges.

In this case, according to the parity of the number of the edges in C_{i_2} and C_3 respectively, it will be discussed in the following two subcases.

Subcase 3.2.2-3a: In graph G^*, C_{i_2} contains an odd number of edges, and C_3 contains an even number of edges.

Because there is exactly one $u, \omega\text{-}path$ in $\mathbb{T}$ for any two vertices u and ω in G, and none of $\{vv_1, vv_2, vv_3\}$ is an edge of $\mathbb{T}$, there must be exactly one $v, v_3\text{-}path$ in $\mathbb{T}$, and the $v, v_3\text{-}path$ in $\mathbb{T}$ must be of the form as $vv_4 \ldots v_3$. Noticing that, in the graph G^*, the connected components C_3 and $C_{i_2} \cup v_{i_2}v'' \cup v''v' \cup v'v_{i_1} \cup C_{i_1}$ both contain an even number of edges, we can easily get that $T^* = (G^* \cap \mathbb{T}) \cup \{v'v_4, v''v_3\}$ is a splitting tree of G^*, which is depicted in Fig.21.

Subcase 3.2.2-3b: In graph G^*, C_{i_2} contains an even number of edges, and C_3 contains an odd number of edges.

In this case, noticing that in the graph G^* the connected components C_{i_2} and $C_3 \cup v_3v'' \cup v''v' \cup v'v_{i_1} \cup C_{i_1}$ both contain an even number of edges, we can easily get that $T^* = (G^* \cap \mathbb{T}) \cup \{v'v_4, v''v_{i_2}\}$ is a splitting tree of G^*, which is depicted in Fig.22.

From Case 1, Case 2, and Case 3, the Lemma 2.2 is obtained. $\qquad \square$

Theorem 2.1 Let G be a graph with minimum degree at least 3, v be a vertex of G with $\deg_G(v) \geqslant 4$, G^* be the graph obtained from G by splitting v into two adjacent vertices v' and v'', furthermore, the $v\text{-}local$ subgraph $G_{loc}(v)$ be connected. Then the graph G is upper embeddable if and only if G^* is upper embeddable.

Proof ($\Longleftarrow$) Let E^* be an embedding of G^* in the orientable surfaces S_g of genus g. Then we can get an embedding E of G in the surface S_g by contracting the *splitting-edge* $v'v''$ in E^*. So $\lfloor \frac{\beta(G)}{2} \rfloor = \lfloor \frac{\beta(G^*)}{2} \rfloor = \gamma_M(G^*) \leqslant \gamma_M(G)$. On the other hand, $\gamma_M(G) \leqslant \lfloor \frac{\beta(G)}{2} \rfloor$. Therefore, $\gamma_M(G) = \lfloor \frac{\beta(G)}{2} \rfloor$, *i.e.*, the graph G is upper embeddable.

($\Longrightarrow$)   Let v_1, v_2, ..., $v_n (n \geqslant 4)$ be all the vertices adjacent to v in G, v' and v'' be the replacement of v after the vertex splitting on v in G, and the edge subset $\{vv_i | i = 1, 2, \ldots n\}$ of $E(G)$ is replaced by the subset $\{v^*v_i | v^*$ may be v' or v'', $i = 1, 2, \ldots n\}$ of $E(G^*)$. It can be obtained from Lemma 2.1 that there exists a splitting tree $\mathbb{T}$ of G such that all of $\{vv_1, vv_2, \ldots, vv_n\}$ are edges of $\mathbb{T}$. Let $T^* = \{G^* \cap \mathbb{T}\} \cup v'v'' \cup \{v^*v_i | v^*$ may be v' or v'', $i = 1, 2, \ldots n\}$. Obviously, T^* is a spanning tree of G^*, and $\xi(G^*, T^*) = \xi(G, \mathbb{T}) = \xi(G) \leqslant 1$. So T^* is a splitting tree of G^*, and G^* is upper embeddable. $\square$

Especially, for a vertex v of G with $\deg_G(v)=4$, we have the following theorem.

Theorem 2.2   Let G be a graph with minimum degree at least 3, v be a vertex of G with $\deg_G(v)=4$, G^* be the graph obtained from G by splitting v into two adjacent vertices v' and v'', where the *splitting-edge* $v'v''$ is not a cut-edge of the *v-splitting* subgraph $G^*_{spl}(v)$. Then the graph G is upper embeddable if and only if G^* is upper embeddable.

Proof   ($\Longleftarrow$)   It is the same with that of the Theorem 2.1.
($\Longrightarrow$)   It is an obvious result of the Lemma 2.2. $\square$

3. Weak minor and upper embeddability

In this section, we will provide a method to construct a weak-minor-closed family of upper embeddable graphs from the bouquet of circles B_n; in addition, we provide a corollary which extends a result obtained by L. Nebeský [17].

Let v be a vertex of the graph G with $\deg_G(v) \geqslant 4$, G^* be the graph obtained from G by splitting v into two adjacent vertices v' and v'', then v is referred to as a *flexible-vertex* of G if it satisfies one of the following two conditions: (I) If v is a vertex of the graph G with $\deg_G(v) \geqslant 4$, then the *v-local* subgraph $G_{loc}(v)$ is connected (and the vertex splitting operation on this kind of vertices is referred to as *type-I vertex splitting*); (II) If v is a vertex of the graph G with $\deg_G(v)=4$, then the *splitting-edge* $v'v''$ is not a cut-edge of the *v-splitting* subgraph $G^*_{spl}(v)$ (this kind of vertex splitting operation is referred to as *type-II vertex splitting*).

According to Theorem 2.1 and Theorem 2.2, we can get, from the bouquet of circles B_n, a weak-minor-closed family of upper embeddable graphs through a sequence of vertex splitting operations on the *flexible-vertices*.

A graph G is called locally connected if for every vertex v of G the *v-local* subgraph $G_{loc}(v)$ is connected. In 1981, L. Nebeský [17] obtained that every connected, locally connected graph is upper embeddable. The following corollary extends this result.

Corollary   A graph, which is obtained from a connected, locally connected graph through a sequence of type-I or type-II vertex splitting operations on it, is upper embeddable.

Proof   According to the result obtained by L. Nebeský [17] we can get that every connected, locally connected graph is upper embeddable. Combining with Theorem 2.1 and Theorem 2.2 we can get the Corollary.   □

4. Conclusions

Remark 1   Let G be an upper embeddable graph with minimum degree at least 3, v be a vertex of G with $\deg_G(v) \geqslant 5$, G^* be the graph obtained from G by splitting v into two adjacent vertices v' and v''. Then the condition that the *splitting-edge* $v'v''$ is not a cut-edge of the *v-splitting* subgraph $G^*_{spl}(v)$ can not guarantee the upper embeddability of G^*. For example, the graph G^* in Fig.24 is a graph obtained from the upper embeddable graph G in Fig.23 through vertex splitting on v in G, and the *splitting-edge* $v'v''$ is not a cut-edge of the *v-splitting* subgraph $G^*_{spl}(v)$. But, G^* is not upper embeddable.
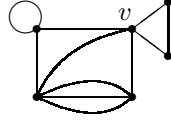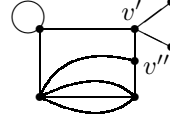

Fig.23: G **Fig.24:** G^*

Remark 2   Let v_1v_2 be an edge of the graph G. The *edge-global subgraph* of v_1v_2, which is denoted by $G_{glo}(v_1v_2)$, is the subgraph of G that is induced by the vertices of v_1, v_2 and all the neighbors of them. The *edge-local subgraph* of v_1v_2, which is denoted by $G_{loc}(v_1v_2)$, is the subgraph of G that is induced by all the neighbors of the vertex v_1 and v_2. A *flexible-edge* of graph G is such an edge v_1v_2 of G which satisfies one of the following two conditions: (I) v_1v_2 is not a cut-edge of the *edge-global subgraph* of v_1v_2, and the adjacent vertices v_1, v_2 are replaced by a vertex v of degree 4 after contracting the edge v_1v_2; (II) The *edge-local subgraph* $G_{loc}(v_1v_2)$ of v_1v_2 is connected, and the adjacent vertices v_1, v_2 are replaced by a vertex v with degree no less than 4 after contracting the edge v_1v_2. A *flexible-weak-minor* of the graph G is a graph obtained from G through a sequence of edge-contraction operations on the *flexible-edges*.

From Theorem 2.1 and Theorem 2.2 we can get that a graph G is upper embeddable if and only if its *flexible-weak-minor* is upper embeddable. So the determining of the upper embeddability of G can be replaced by determining the upper embeddability of its *flexible-weak-minor*. Furthermore, the algorithm complexity of determining the upper embeddability of G may be reduced much by this way, because the order of the *flexible-weak-minor* of G is less than the order of G.

References

[1] D.B. West, Introduction to Graph Theory, Prentice Hall, Upper Saddle River, NJ, 2001.

[2] B. Mohar and C. Thomassen, Graphs on Surfaces, Johns Hopkins University Press, 2001.

13

[3] J.L. Gross, and T.W. Tucker, Topological graph theory. Wiley-Interscience, New York, 1987.

[4] Y.P. Liu, Embeddability in Graphs, Kluwer Academic, Dordrecht, Boston, London, 1995.

[5] E.A. Nordhause, B.M. Stewart, A.T. White, On the maximum genus of a graph, J. Combin. Theory, 11 (1971) 258−267.

[6] C. Thomassen, Bidirectional retracting-free double tracings and upper embeddability of graphs, J. Combin. Theory Ser. B, 50 (1990) 198-207.

[7] J. Chen, S.P. Kanchi, and J.L. Gross, A tight lower bound on the maximum genus of a simplicial graph, Discrete Math., 156 (1996) 83-102.

[8] Y.Q. Huang, Y.P. Liu, Face size and the maximum genus of a graph, J. Combin. Theory Ser. B, 80 (2000) 356-370.

[9] Z.D. Ouyang, L. Tang, and Y.Q. Huang, Upper embeddability, edge independence number and girth, Science China Math., 52(9) (2009) 1939C1946.

[10] H. Ren, H.T. Zhao, and H.L. Li, Fundamental cycles and graph embeddings, Science in China Ser. A, 52(9) (2009) 1920-1926.

[11] Y.C. Chen, Y.P. Liu. Maximum genus, girth and maximum non-adjacent edge set, Ars Combin., 79 (2006) 145C159.

[12] D.M. Li and Y.P. Liu, Maximum genus, girth and connectivity, European. J. Combin. 21 (2000) 651-657.

[13] L. Nebeský, A new characterization of the maximum genus of a graph, Czechoslova Math. J., 31(106) (1981) 604-613.

[14] M. Škoviera, The maximum genus of graphs diameter two, Discrete Math., 87 (1991) 175−180.

[15] N.H. Xuong, How to determine the maximum genus of a graph, J. Combin. Theory Ser. B, 26 (1979) 217−225

[16] Y.P. Liu, The maximum orientable genus of a graph, Scientia Sinical (Special Issue II) (1979) 41-55.

[17] L. Nebeský, Every connected, locally connected graph is upper embeddable, J. Graph Theory, 2(5) (1981) 205-207.